1.7. Partial Fractions

Given a set of fractions, we can add them together to form a single fraction. For example, in Example 1.32 we saw

$$\frac{2}{x+1}+\frac{4}{x+2}=\frac{2(x+2)+4(x+1)}{(x+1)(x+2)}$$
$$=\frac{6x+8}{x^2+3x+2}$$

Alternatively, if we are given a single fraction, we can break it down into the sum of easier fractions. These simple fractions, which when added together form the given fraction, are called partial fractions. The partial fractions of

$$\frac{6x+8}{x^2+3x+2} \text{ are } \frac{2}{x+1} \text{ and } \frac{4}{x+2}.$$

Linear factors

A linear factor $ax+b$ in the denominator produces a partial fraction of the form $\frac{A}{ax+b}$.

Repeated linear factor

Attention of word factor

A repeated linear factor, $(ax+b)^2$, leads to partial fractions

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

Example

$$\frac{2x+5}{x^2+2x+1}=\frac{2x+5}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$$

Multiplying by $(x+1)^2$

$$\frac{2x+5}{x^2+2x+1} = \frac{2}{x+1} + \frac{3}{(x+1)^2}$$

Example

$$\frac{14x^2+13x}{(4x^2+4x+1)(x-1)} = \frac{14x^2+13x}{(2x+1)^2(x-1)} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-1}$$

$$\frac{14x^2+13x}{(4x^2+4x+1)(x-1)} = \frac{1}{2x+1} + \frac{2}{(2x+1)^2} + \frac{3}{x-1}$$

Quadratic factors

Exercise

Express the following as partial fractions.

$$\frac{13x^2+11x+2}{(x+1)(2x+1)(3x+1)}$$

Solutions

$$\frac{13x^2+11x+2}{(x+1)(2x+1)(3x+1)} = \frac{A}{x+1} + \frac{B}{2x+1} + \frac{C}{3x+1}$$

$$13x^2+11x+2 = A(2x+1)(3x+1) + B(x+1)(3x+1) + C(x+1)(2x+1)$$

1) if $x = -\frac{1}{2}$, Left side $= -\frac{1}{4}$, Right side $= -\frac{1}{4}B$

$\therefore B = 1$

2) if $x = -\frac{1}{3}$, Left side $= -\frac{2}{9}$, Right side $= \frac{2}{9}C$

$\therefore C = -1$

2) if $x = -1$, Left side = 4, Right side = 2A

$\therefore A = 2$

$$\frac{13x^2 + 11x + 2}{(x+1)(2x+1)(3x+1)} = \frac{2}{x+1} + \frac{1}{2x+1} - \frac{1}{3x+1}$$

$$\frac{3x^2 + 8x + 6}{(x^2 + 2x + 1)(x+2)}$$

Solutions

$$\frac{3x^2 + 8x + 6}{(x^2 + 2x + 1)(x+2)} = \frac{3x^2 + 8x + 6}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

multiplying both by $(x+1)^2(x+2)$

$3x^2 + 8x + 6 = A(x+1)(x+2) + B(x+2) + c(x+1)^2$

1) $x = -1,\ L = 1, R = B \therefore B = 1$

2) $x = -2,\ L = 2, R = C \therefore C = 2$

3) $x = 1,\ L = 17, R = 6A + 1 \therefore A = 1$

$$\frac{3x^2 + 8x + 6}{(x^2 + 2x + 1)(x+2)} = \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{x+2}$$